Problem Set 2, Solutions

1 Exercise, OLG (1)

Suppose that N_t 2-period-lived (identical) individuals are born in period t and that $N_{t+1} = (1+n) N_t$. For simplicity, let utility be logarithmic: $U_t = \ln(C_{1,t}) + \beta \ln(C_{2,t+1})$. A subscript 1 (2) denotes that the variable pertains to the young (old) generation. The economy is closed and capital markets are perfect. Each individual is endowed with Y_1 when young and with Y_2 when old.

1. Show that the number of young alive is a constant fraction of total population.

   Solution 1 *Total population in* $t+1$ *is given by* $N_{t+1} + N_t$. *We also know that* $N_t = (1+n) N_{t-1}$ *and* $N_{t+1} = (1+n) N_t$. *Combining we get*

$$\begin{aligned} N_{t+1} + N_t &= (1+n) N_{t-1} + (1+n) N_t \\ (1+n) N_t + N_t &= (1+n)(N_{t-1} + N_t) \\ N_t &= \frac{(1+n)}{(2+n)}(N_{t-1} + N_t) \end{aligned}$$

   Note that if there were no population growth ($n = 0$), $N_t = \frac{1}{2}(N_{t-1} + N_t)$, *half of the population would be young half would be old.*

2. The utility function is concave, the household, therefore, prefers to smooth consumption. In this model there are four possible ways the household could smooth consumption (one more than in the Fisher model). What are these?

   Solution 2

   1) storage (but good is usually assumed perishable)

   2) foreign trade

   3) trade within cohort (generation)

   4) trade across cohort

3. From now on we will assume that the good is not storable and that the economy is closed. Will there be trade within the same generation? Will there be trade between generations?

Solution 3 *Since we assumed that individuals are identical there cannot be trade within the same generation. This is the same problem as in the Fisher model. In theory trade between generations would be possible, but here we assumed that a generation only lives for two periods. Trade between young and old in period t cannot be completed because the old do not exist anymore in period $t+1$.*

4. In a decentralized economy, write down the household's budget constraint and derive the equilibrium interest rate in terms of exogenous variables.

Solution 4 *The budget constraint is, as in the Fisher model given by*

$$C_{1,t} + \frac{C_{2,t+1}}{1+r} = Y_1 + \frac{Y_2}{1+r}$$

To derive the equilibrium interest rate we need to derive the consumption Euler equation. The household problem is to

$$\max_{C_{1t}, C_{2,t+1}} \ln(C_{1,t}) + \beta \ln(C_{2,t+1})$$

$$s.t. C_{1,t} + \frac{C_{2,t+1}}{1+r} = Y_1 + \frac{Y_2}{1+r}$$

The first order conditions are

$$\frac{1}{C_{1,t}} = \lambda$$

$$\beta \frac{1}{C_{2,t+1}} = \frac{\lambda}{1+r}$$

Combining we get the consumption Euler equation $C_{2,t+1} = C_{1,t}\beta(1+r)$. In a closed economy the consumption point has to equal the endowment point, i.e. $(C_{1,t}, C_{2,t+1}) = (Y_1, Y_2)$. The equilibrium interest rate can then be written as

$$(1+r) = \frac{C_{2,t+1}}{C_{1,t}\beta} = \frac{Y_2}{\beta Y_1}$$

The equilibrium interest rate assures that the market for bonds is in equilibrium, here, this means that $B_t = 0, \forall t$.

5. Show the decentralized equilibrium graphically in the C_1, C_2 space. In particular, show the consumption point and the endowment point. Note that the economy is stationary, i.e. consumption of the old or of the young is the same regardless of the generation. C_2 (on the vertical axis), therefore, stands for both the consumption of the old in t and of the young in $t+1$ (when they are old).

Solution 5 *...*

6. There are situations where the decentralized equilibrium is not pareto optimal. Give an intuition why in the OLG model a social planner can improve the household's welfare and in the Fisher model not.

   Solution 6 *A social planner can improve welfare if he opens trading possibilities that are not present in a decentralized economy. The difference between the Fisher and the OLG model is that in the (closed economy) Fisher model there is simply no one to trade with, while in the OLG at each point in time there are 2 generations alive. In principle goods could be exchanged between these two generations but the fact the each generation only lives for two periods prohibits any decentralized trade between generations.*

7. Derive the social planner's budget constraint. Hint: total consumption in t must equal total endowment in t.

   Solution 7 *Total consumption is given by* $N_t C_{1,t} + N_{t-1} C_{2,t}$ *and total endowment is given by* $N_t Y_1 + N_{t-1} Y_2$*. We get, therefore*

$$\begin{aligned} N_t C_{1,t} + N_{t-1} C_{2,t} &= N_t Y_1 + N_{t-1} Y_2 \\ C_{1,t} + \frac{N_{t-1}}{N_t} C_{2,t} &= Y_1 + \frac{N_{t-1}}{N_t} Y_2 \\ C_{1,t} + C_{2,t} \frac{1}{1+n} &= Y_1 + \frac{1}{1+n} Y_2 \end{aligned}$$

8. Assume that the social planner can impose a lump sum tax τ (subsidy if negative) to redistribute the consumption good from one generation to another. Per capita consumption of the young then becomes

$$C_{1,t} = Y_1 - \tau$$

   and per capita consumption of the old

$$C_{2,t} = Y_2 + \tau (1+n)$$

   The objective of the social planner is to maximize utility subject to feasibility. The choice variable of the social planner is τ. Set up the social planner's maximization problem.

   Solution 8 *The problem of the social planner is to maximize*

$$\begin{aligned} & \max_{\tau} u(C_1) + \beta u(C_2) \\ s.t. C_1 &= Y_1 - \tau \\ C_2 &= Y_2 + \tau (1+n) \end{aligned}$$

   Where we dropped the time subscripts on C_1 *and* C_2*.*

9. Derive the first order condition of the planner's problem and show that $\tau > 0$ if $n > r$.

Solution 9

$$\begin{aligned} \max_{\tau} u(Y_1 - \tau) + \beta u(Y_2 + \tau(1+n)) &\quad : \\ -u'(Y_1 - \tau) + \beta(1+n) u'(Y_2 + \tau(1+n)) &= 0 \end{aligned}$$

rewriting this we get

$$(1+n) = \frac{u'(C_1)}{\beta u'(C_2)} = \frac{u'(Y_1 - \tau)}{\beta u'(Y_2 + \tau(1+n))}$$

If $n > r$ we find that

$$(1+n) = \frac{u'(Y_1 - \tau)}{\beta u'(Y_2 + \tau(1+n))} > \frac{u'(Y_1)}{\beta u'(Y_2)} = (1+r)$$

For this to hold τ has to be positive.

2 Exercise OLG(2)

Suppose that N_t 2-period-lived individuals are born in t and that $N_t = (1+n) N_{t-1}$. For simplicity, let utility be logarithmic with no discouting: $U_t = \ln(C_{1,t}) + \ln(C_{2,t+1})$. Each individual born at time t is endowed with A units of the economy's single good. The good can either be consumed or stored. Each unit stored yields $x > 0$ units of the good in the following period.

Finally assume that in the initial period, period 0, in addition to the N_0 young individuals each endowed with A units of the good, there are $\frac{1}{1+n} N_0$ individuals who are alive only in period 0. Each of these "old" individual's is endowed with some amount Z of the good; their utility is simply their consumption in the initial period, C_{20}.

1. Describe the decentralized equilibrium of this economy. (Hint: given the overlapping-generations structure, will the members of any generation engage in transactions with members of another generation?

Solution 10 *In the decentralized equilibrium there will be no intergenerational trade. Even if the young would like to trade goods this period for goods next period, the only people around to trade with are the old. Unfortunately, the old will have died in the next period - and thus in no position to complete the trade - next period.*

The individual's utility function is given by

$$\ln C_{1,t} + \ln C_{2,t+1}$$

The constraints are

$$\begin{aligned} C_{1,t} + F_t &= A \\ \textit{and} \quad C_{2,t+1} &= xF_t \end{aligned}$$

where F_t is the amount stored by the individual. From the equations above we get the intertemporal budget constraint.

$$C_{1,t} + \frac{C_{2,t+1}}{x} = A$$

The individual's problem is to maximize lifetime utility subject to the intertemporal budget constraint. Set up the Lagrangian

$$L = \ln C_{1,t} + \ln C_{2,t+1} + \lambda \left(A - C_{1,t} - C_{2,t+1}/x\right)$$

The first order conditions are given by:

$$\begin{aligned} \frac{\partial L}{\partial C_{1,t}} &= \frac{1}{C_{1,t}} - \lambda \\ \frac{\partial L}{\partial C_{2,t+1}} &= \frac{1}{C_{2,t+1}} - \frac{\lambda}{x} \end{aligned}$$

Combining we get the consumption Euler equation

$$C_{2,t+1} = xC_{1,t}$$

Substitute the Euler equation in the budget constraint to get

$$C_{1,t} = \frac{A}{2} \textit{ and } C_{2,t+1} = \frac{xA}{2}$$

When young, each individual consumes half of her endowment and stores the other half. This allows her to consumer $\frac{xA}{2}$ when old. Note that with log utility, the fraction of endowment the individual stores is independent of the return on storage (x).

2. Consider paths where the fraction of agent's endowments that is stored, f_t, is constant over time. What is total consumption (that is, consumption of all the young plus consumption of all the old) per person on such a path as a function of f? If $x < 1+n$ what value of f satisfying $0 \leq f \leq 1$ maximizes consumption per person? Is the decentralized equilibrium Pareto-efficient in this case? If not, how can a social planner raise welfare?

Solution 11 *What is consumption per unit of effective labour at time t? First calculate total consumption*

$$C_t = C_{1,t}N_t + C_{2,t}N_{t-1}$$

where N_t *young and* N_{t-1} *old people are alive at time* t. *Each young person consumes the fraction of their endowment they do not store.* $(1-f)\,A$, *while each old person gets to consume the part they they stored. Thus*

$$C_t = (1-f)\,AN_t + fxAN_{t-1}$$

To convert this into units of effective labour, divide both sides by AN_t *to get*

$$\frac{C_t}{AN_t} = (1-f) + f\left(\frac{x}{1+n}\right)$$

Thus, consumption per unit of time t effective labour is a weighted average of 1 *and something less than 1 since* $x < (1+n)$. *It will therefore be maximized when weight on 1 is 1, that is, when* $f = 0$.

The decentralized equilibrium, with $f = \frac{1}{2}$ *is not Pareto efficient. Since intergenerational trade is not possible, people are "forced" into storage because that is the only way they can save and consume at the old age. They must do this even if the return on storage, x, is low. However, at any point in time, a social planner could take 1 unit from each young and give* $(1+n)$ *units to each old person since there are fewer of them. With* $(1+n) > x$, *this gives a better return than storage. Therefore, the social planner could raise welfare by taking the half of each generation's endowment that they were going to store and give it to the old.The planner could then do this each period. This allows people to still consume* $A/2$ *when young and now* $(1+n)\,A/2$ *when old. This is greater than the* $xA/2$ *units of consumption when old that they would have had in the decentralized equilibrium with storage.*

Also note that the first generation is better off. A social planner, can, therefore Pareto-improve the decentralized equilibrium.